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**LEXICOGRAPHIC OPTIMIZATION ON POLYTOPES IS LINEAR
PROGRAMMING**

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Lexicographic Optimization on Polytopes is Linear Programming

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Abstract

Finding the lexicographic maximum of a polytope in \mathbb{R}^n can be achieved by solving a suitable LP-problem.

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1 Introduction

Recently, the average lexicographic value for balanced cooperative games has been introduced by Tijs (2005). This value assigns to each balanced game the average of lexicographic maxima of the core of a game. Since the core of a game is a polytope the question arises whether it is easier to calculate the lexicographic maximum for a polytope than for an arbitrary compact set in \mathbb{R}^n . In this note we show that for a polytope the lexicographic maximum can be obtained by linear programming (LP).

2 The Result

In \mathbb{R}^n the lexicographic order \geq_L is defined as follows. For $x, y \in \mathbb{R}^n$ we have $x \geq_L y$ if $x = y$ or if there is an $r \in \{1, 2, \dots, n\}$ such that $x_k = y_k$ for $k < r$ and $x_r > y_r$. It is well-known that for a compact set C in \mathbb{R}^n there exists a unique lexicographic maximum, denoted here by $\text{lexmax}(C)$, which is the point in C , which is lexicographic larger than each other point in C . If C is also convex, then $\text{lexmax}(C)$ is an extreme point of C . In general, finding $\text{lexmax}(C)$ can be achieved by solving n classical optimization problems. The following theorem shows that one optimization problem suffices in case C is a polytope. Since a polytope is a polyhedral set the optimization problem is in fact an LP-problem.

Theorem: Let P be a polytope in \mathbb{R}^n and let L be the lexicographic maximum of P . Then, for small $\varepsilon > 0$, L is the unique point in P , where the inner product of $e^\varepsilon = (1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1})$ with $x \in P$ is maximal.

Proof: Let $\text{ext}(P)$ be the finite set of extreme points of P . Take $z \in \text{ext}(P) \setminus \{L\}$ and let $r \in \{1, 2, \dots, n\}$ be such that $L_k = z_k$ for $k < r$ and $L_r > z_r$. For $\varepsilon > 0$, $(e^\varepsilon)^T(L - z) = \varepsilon^{r-1}(L_r - z_r) + \sum_{k=r}^{n-1} \varepsilon^k(L_{k+1} - z_{k+1}) = \varepsilon^{r-1}a(\varepsilon)$ with $a(\varepsilon) = (L_r - z_r) + \sum_{k=r}^{n-1} \varepsilon^{k-r+1}(L_{k+1} - z_{k+1})$ and $\lim_{\varepsilon \downarrow 0} a(\varepsilon) = L_r - z_r > 0$. This implies that there is an $\varepsilon(z) > 0$ such that $(e^\varepsilon)^T(L - z) > 0$ for all $\varepsilon \leq \varepsilon(z)$.

Take $\hat{\varepsilon} = \min \{\varepsilon(z) \mid z \in \text{ext}(P) \setminus \{L\}\}$. Then for all $\varepsilon \leq \hat{\varepsilon}$

$$(e^\varepsilon)^T(L - z) > 0 \text{ for all } z \in \text{ext}(P) \setminus \{L\}.$$

$$\text{So, } L = \text{argmax} \left\{ (e^\varepsilon)^T z \mid z \in \text{ext}(P) \right\}$$

$$= \operatorname{argmax} \left\{ (e^\varepsilon)^T x \mid x \in P \right\} \text{ for all } \varepsilon \leq \hat{\varepsilon}. \quad \square$$

Reference

Tijs, S.H., The first steps with Alexia, the average lexicographic value. CentER Discussion Paper 2005 - 123, 2005, Tilburg University, The Netherlands